

LOCALISING SUBCATEGORIES FOR COCHAINS ON THE CLASSIFYING SPACE OF A FINITE GROUP

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ABSTRACT. The localising subcategories of the derived category of the cochains on the classifying space of a finite group are classified. They are in one to one correspondence with the subsets of the set of homogeneous prime ideals of the cohomology ring $H^*(G, k)$.

1. INTRODUCTION

Let G be a finite group and k a field of characteristic p . Let $C^*(BG; k)$ be the cochains on the classifying space BG . Using the machinery of Elmendorf, Kříž, Mandell and May [8], one can regard $C^*(BG; k)$ as a strictly commutative S -algebra over the field k . The derived category $D(C^*(BG; k))$ has thus a structure of a tensor triangulated category via the left derived tensor product $- \otimes_{C^*(BG; k)}^{\mathbf{L}} -$. The unit for the tensor product is $C^*(BG; k)$.

In this paper we apply techniques and results from [3, 4, 5, 6] to classify the localising subcategories of $D(C^*(BG; k))$. More precisely, there is a notion of stratification for triangulated categories via the action of a graded commutative ring which implies that the localising subcategories are parameterised by sets of homogeneous prime ideals [4]. For $D(C^*(BG; k))$ we use the natural action of the endomorphism ring of the tensor identity which is isomorphic to the cohomology algebra $H^*(G, k)$ of the group G .

Theorem 1.1. *The derived category $D(C^*(BG; k))$ is stratified by the ring $H^*(G, k)$. This yields a one to one correspondence between the localising subcategories of $D(C^*(BG; k))$ and subsets of the set of homogeneous prime ideals of $H^*(G, k)$.*

It is proved in [6] that there is an equivalence of tensor triangulated categories between $D(C^*(BG; k))$ and the localising subcategory of $K(\text{Inj } kG)$ generated by the tensor identity. Here, $K(\text{Inj } kG)$ is the homotopy category of complexes of injective (= projective) kG -modules, studied in [6, 9].

The main theorem of [5] states that $K(\text{Inj } kG)$ is stratified as a tensor triangulated category by $H^*(G, k)$. Theorem 1.1 is a consequence of a more general result concerning tensor triangulated categories, which is described below.

Let $(T, \otimes, 1)$ be a compactly generated tensor triangulated category, as described in [3, §8], and R a graded commutative noetherian ring acting on T via a homomorphism $R \rightarrow \text{End}_T^*(1)$. In this case, for each homogeneous prime ideal \mathfrak{p} of R there exists a *local cohomology functor* $\Gamma_{\mathfrak{p}}: T \rightarrow T$; see [3]. The *support* of an object X in T is then defined to be

$$\text{supp}_R X = \{\mathfrak{p} \in \text{Spec } R \mid \Gamma_{\mathfrak{p}} X \neq 0\}.$$

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The condition that T is *stratified* by the action of R means that assigning a subcategory S of T to its support

$$\mathrm{supp}_R \mathsf{S} = \bigcup_{X \in \mathsf{S}} \mathrm{supp}_R X$$

yields a bijection between *tensor ideal* localising subcategories of T and subsets of the homogeneous prime ideal spectrum $\mathrm{Spec} R$ contained in $\mathrm{supp}_R \mathsf{T}$; see [4, Theorem 4.2]. Theorem 1.1 is thus a special case of the result below that relates tensor ideal localising subcategories of T and the localising subcategories of $\mathrm{Loc}_{\mathsf{T}}(\mathbb{1})$, the localising subcategory of T generated by the tensor unit. We note that $\mathrm{Loc}_{\mathsf{T}}(\mathbb{1})$ is a compactly generated tensor triangulated category in its own right and that R acts on it as well.

Theorem 1.2. *Suppose that the Krull dimension of R is finite. If T is stratified by R as a tensor triangulated category, then so is $\mathrm{Loc}_{\mathsf{T}}(\mathbb{1})$, and there is a bijection*

$$\left\{ \begin{array}{c} \text{Tensor ideal localising} \\ \text{subcategories of } \mathsf{T} \end{array} \right\} \xrightarrow{\sim} \left\{ \begin{array}{c} \text{Localising subcategories} \\ \text{of } \mathrm{Loc}_{\mathsf{T}}(\mathbb{1}) \end{array} \right\}.$$

It assigns each tensor ideal localising subcategory S of T to $\mathsf{S} \cap \mathrm{Loc}_{\mathsf{T}}(\mathbb{1})$.

Remark 1.3. The theorem is not true without the assumption that T is stratified by R . For example, let T be the derived category of quasi-coherent sheaves on the projective line \mathbb{P}_k^1 . The tensor unit is \mathcal{O} . In this example there are no proper localising subcategories of $\mathrm{Loc}_{\mathsf{T}}(\mathcal{O})$ since $\mathrm{End}_{\mathsf{T}}^*(\mathcal{O}) = k$, while there are many tensor ideal localising subcategories of T .

Remark 1.4. The assumption that the Krull dimension of R is finite is artificial, and is used only to ensure that for each $X \in \mathsf{T}$ and $\mathfrak{p} \in \mathrm{Spec} R$ the object $\Gamma_{\mathfrak{p}} X$ belongs to $\mathrm{Loc}_{\mathsf{T}}(X)$. One can replace this condition by, for instance, the assumption that T arises as the homotopy category of a Quillen model category [10, §6].

2. LOCALISING SUBCATEGORIES OF $\mathrm{Loc}_{\mathsf{T}}(\mathbb{1})$

In this section T is a triangulated category with set-indexed coproducts and the tensor product \otimes provides a symmetric monoidal structure with unit $\mathbb{1}$ on T , which is exact in each variable and preserves set-indexed coproducts.

The proof of Theorem 1.2 is based on a sequence of elementary lemmas. The first one describes the tensor ideal localising subcategory of T which is generated by a class C of objects; we denote this by $\mathrm{Loc}_{\mathsf{T}}^{\otimes}(\mathsf{C})$.

Lemma 2.1. *Let C be a class of objects of T . Then*

$$\mathrm{Loc}_{\mathsf{T}}^{\otimes}(\mathsf{C}) = \mathrm{Loc}_{\mathsf{T}}(\{X \otimes Y \mid X \in \mathsf{C}, Y \in \mathsf{T}\}).$$

Proof. Set $\mathsf{S} = \mathrm{Loc}_{\mathsf{T}}(\{X \otimes Y \mid X \in \mathsf{C}, Y \in \mathsf{T}\})$. It suffices to show that S is tensor ideal. This means that $F\mathsf{S} \subseteq \mathsf{S}$ for each tensor functor $F = - \otimes Y$, which is an immediate consequence of Lemma 2.2 below. \square

Lemma 2.2. *Let $F: \mathsf{U} \rightarrow \mathsf{V}$ be an exact functor between triangulated categories that preserves set-indexed coproducts. If C is a class of objects of U , then*

$$F \mathrm{Loc}_{\mathsf{U}}(\mathsf{C}) \subseteq \mathrm{Loc}_{\mathsf{V}}(F\mathsf{C}).$$

Proof. The preimage $F^{-1} \mathrm{Loc}_{\mathsf{V}}(F\mathsf{C})$ is a localising subcategory of U containing C . Thus it contains $\mathrm{Loc}_{\mathsf{U}}(\mathsf{C})$, and one gets

$$F \mathrm{Loc}_{\mathsf{U}}(\mathsf{C}) \subseteq FF^{-1} \mathrm{Loc}_{\mathsf{V}}(F\mathsf{C}) \subseteq \mathrm{Loc}_{\mathsf{V}}(F\mathsf{C}). \quad \square$$

Lemma 2.3. *Let $\Gamma: \mathcal{T} \rightarrow \mathcal{T}$ be a colocalization functor that preserves set-indexed coproducts. Then for any $X \in \mathcal{T}$ and $Y \in \text{Loc}_{\mathcal{T}}(\mathbb{1})$, there is a natural isomorphism*

$$\Gamma X \otimes Y \xrightarrow{\sim} \Gamma(X \otimes Y).$$

Remark 2.4. There is an analogous result for a localization functor $L: \mathcal{T} \rightarrow \mathcal{T}$ that preserves set-indexed coproducts: For any $X \in \mathcal{T}$ and $Y \in \text{Loc}_{\mathcal{T}}(\mathbb{1})$, there is a natural isomorphism $L(X \otimes Y) \xrightarrow{\sim} LX \otimes Y$.

Proof. A colocalisation functor Γ comes with a natural morphism $\Gamma X \rightarrow X$. Tensoring this with an object $Y \in \text{Loc}_{\mathcal{T}}(\mathbb{1})$ gives a morphism $\Gamma X \otimes Y \rightarrow X \otimes Y$ that factors through the natural morphism $\Gamma(X \otimes Y) \rightarrow X \otimes Y$. Here, one uses that $\Gamma X \otimes Y$ belongs to $\Gamma\mathcal{T}$, since the objects $Y' \in \mathcal{T}$ with $\Gamma X \otimes Y' \in \Gamma\mathcal{T}$ form a localising subcategory containing $\mathbb{1}$. The induced morphism $\phi_Y: \Gamma X \otimes Y \rightarrow \Gamma(X \otimes Y)$ is an isomorphism. To see this, observe that the objects $Y' \in \mathcal{T}$ such that $\phi_{Y'}$ is an isomorphism form a localising subcategory containing $\mathbb{1}$. \square

Proposition 2.5. *Suppose that the unit $\mathbb{1}$ is compact in \mathcal{T} and let $\Gamma: \mathcal{T} \rightarrow \text{Loc}_{\mathcal{T}}(\mathbb{1})$ denote the right adjoint of the inclusion $\text{Loc}_{\mathcal{T}}(\mathbb{1}) \rightarrow \mathcal{T}$. If S is a localising subcategory of $\text{Loc}_{\mathcal{T}}(\mathbb{1})$, then*

$$\text{Loc}_{\mathcal{T}}^{\otimes}(S) \cap \text{Loc}_{\mathcal{T}}(\mathbb{1}) = \Gamma(\text{Loc}_{\mathcal{T}}^{\otimes}(S)) = S.$$

Proof. We verify each of the following inclusions

$$S \subseteq \text{Loc}_{\mathcal{T}}^{\otimes}(S) \cap \text{Loc}_{\mathcal{T}}(\mathbb{1}) \subseteq \Gamma(\text{Loc}_{\mathcal{T}}^{\otimes}(S)) \subseteq S.$$

The first one is clear. Composing the functor Γ with the inclusion $\text{Loc}_{\mathcal{T}}(\mathbb{1}) \rightarrow \mathcal{T}$ yields a colocalisation functor that preserves set-indexed coproducts, since $\mathbb{1}$ is compact. For an object X in $\text{Loc}_{\mathcal{T}}^{\otimes}(S) \cap \text{Loc}_{\mathcal{T}}(\mathbb{1})$, we have $\Gamma X \cong X$. This gives the second inclusion. Applying Lemma 2.3 together with the description of $\text{Loc}_{\mathcal{T}}^{\otimes}(S)$ from Lemma 2.1 yields the third inclusion. \square

Corollary 2.6. *Suppose that the unit $\mathbb{1}$ is a compact object in \mathcal{T} . Assigning each localising subcategory S of $\text{Loc}_{\mathcal{T}}(\mathbb{1})$ to $\text{Loc}_{\mathcal{T}}^{\otimes}(S)$ gives a bijection*

$$\left\{ \begin{array}{c} \text{Localising subcategories} \\ \text{of } \text{Loc}_{\mathcal{T}}(\mathbb{1}) \end{array} \right\} \xrightarrow{\sim} \left\{ \begin{array}{c} \text{Tensor ideal localising subcategories of} \\ \mathcal{T} \text{ generated by objects from } \text{Loc}_{\mathcal{T}}(\mathbb{1}) \end{array} \right\}.$$

Proof. The inverse map sends $U \subseteq \mathcal{T}$ to $U \cap \text{Loc}_{\mathcal{T}}(\mathbb{1})$. \square

We are now ready to prove Theorem 1.2. Note that in this \mathcal{T} is a compactly generated tensor triangulated category, which entails a host of additional requirements; see [3, §8] for a list.

Proof of Theorem 1.2. It follows from Proposition 2.5 that the assignment

$$S \mapsto \text{Loc}_{\mathcal{T}}^{\otimes}(S)$$

is an injective map from the localising subcategories of $\text{Loc}_{\mathcal{T}}(\mathbb{1})$ to the tensor ideal localising subcategories of \mathcal{T} . In general, it is not bijective, as the example of Remark 1.3 shows. However, since \mathcal{T} is stratified by R as a tensor triangulated category, it follows from [4, §7] that each tensor ideal localising subcategory is generated by a set of objects of the form $\Gamma_{\mathfrak{p}}\mathbb{1}$. Since R has finite Krull dimension, [4, Theorem 3.4] yields that $\Gamma_{\mathfrak{p}}\mathbb{1}$ is in $\text{Loc}_{\mathcal{T}}(\mathbb{1})$. Therefore, given a tensor ideal localising subcategory U of \mathcal{T} , the localising subcategory

$$U' = \text{Loc}_{\mathcal{T}}(\{\Gamma_{\mathfrak{p}}\mathbb{1} \mid \mathfrak{p} \in \text{Supp}_R U\}) \subseteq \text{Loc}_{\mathcal{T}}(\mathbb{1})$$

satisfies $\text{Loc}_T^\otimes(U') = U$. This proves the surjectivity of the assignment. Moreover, we have shown that each localising subcategory of $\text{Loc}_T(\mathbb{1})$ is generated by objects of the form $\Gamma_{\mathfrak{p}}\mathbb{1}$, so $\text{Loc}_T(\mathbb{1})$ is stratified by the action of R ; see [4, Theorem 4.2]. \square

3. THE COHOMOLOGICAL NUCLEUS

Let $(T, \otimes, \mathbb{1})$ be a compactly generated tensor triangulated category and let R be a graded commutative noetherian ring acting on T via a homomorphism $R \rightarrow \text{End}_T^*(\mathbb{1})$. Suppose in addition that R has finite Krull dimension.

We define the *cohomological nucleus* of T as the set of homogeneous prime ideals \mathfrak{p} of R such that there exists an object $X \in T$ satisfying $\text{Hom}_T^*(\mathbb{1}, X) = 0$ and $\Gamma_{\mathfrak{p}}X \neq 0$. This definition is motivated by work of Benson, Carlson, and Robinson in the context of modular group representations [2].

For \mathfrak{p} in $\text{Spec } R$ consider the tensor ideal localising subcategory

$$\Gamma_{\mathfrak{p}}T = \{Y \in T \mid Y \cong \Gamma_{\mathfrak{p}}X \text{ for some } X \in T\}.$$

Note that an object $X \in T$ belongs to $\Gamma_{\mathfrak{p}}T$ if and only if $\text{Hom}_T^*(C, X)$ is \mathfrak{p} -local and \mathfrak{p} -torsion for every compact $C \in T$, by [3, Corollary 4.10]. The result below gives a local description of the cohomological nucleus.

Proposition 3.1. *Let \mathfrak{p} be a homogeneous prime ideal of R . The following conditions are equivalent:*

- (1) *Every object X in T with $\text{Hom}_T^*(\mathbb{1}, X) = 0$ satisfies $\Gamma_{\mathfrak{p}}X = 0$.*
- (2) *One has $\text{Loc}_T(\Gamma_{\mathfrak{p}}\mathbb{1}) = \Gamma_{\mathfrak{p}}T$.*
- (3) *Every localising subcategory of $\Gamma_{\mathfrak{p}}T$ is a tensor ideal of T .*

Proof. The Krull dimension of R is finite, so $\Gamma_{\mathfrak{p}}X$ is in $\text{Loc}_T(X)$ for each X in T , by [4, Theorem 3.4]. This fact is used without further comment.

(1) \Rightarrow (2): Set $S = \text{Loc}_T(\Gamma_{\mathfrak{p}}\mathbb{1})$. Note that $S \subseteq \Gamma_{\mathfrak{p}}T$; we claim that equality holds. Indeed, $S \subseteq \text{Loc}_T(\mathbb{1})$ and also $\text{Loc}_T^\otimes(S) = \Gamma_{\mathfrak{p}}T$, since $\Gamma_{\mathfrak{p}} = \Gamma_{\mathfrak{p}}\mathbb{1} \otimes -$. Thus, for any X in $\Gamma_{\mathfrak{p}}T$ from Proposition 2.5 one gets an exact triangle $\Gamma X \rightarrow X \rightarrow X' \rightarrow$ with $\Gamma X \in S$ and $\text{Hom}_T^*(\mathbb{1}, X) = 0$. Then (1) implies $X' = 0$ and hence $X \in S$.

(2) \Rightarrow (3): Let S be a localising subcategory of $\Gamma_{\mathfrak{p}}T$. Using (2) and the fact that $\Gamma_{\mathfrak{p}}T$ is a tensor ideal of T , one has $\text{Loc}_T^\otimes(S) \subseteq \text{Loc}_T(\mathbb{1})$. Then it follows, again from Proposition 2.5, that S is a tensor ideal of T .

(3) \Rightarrow (1): Assume $\text{Hom}_T^*(\mathbb{1}, X) = 0$; then $\text{Hom}_T^*(\mathbb{1}, \Gamma_{\mathfrak{p}}X) = 0$, as $\mathbb{1}$ is compact. Condition (3) implies that $\text{Loc}_T(\Gamma_{\mathfrak{p}}\mathbb{1}) = \Gamma_{\mathfrak{p}}T$. Thus $\Gamma_{\mathfrak{p}}X$ belongs to $\text{Loc}_T(\Gamma_{\mathfrak{p}}\mathbb{1})$ and therefore also to $\text{Loc}_T(\mathbb{1})$. So one obtains $\text{Hom}_T^*(\Gamma_{\mathfrak{p}}X, \Gamma_{\mathfrak{p}}X) = 0$, which implies $\Gamma_{\mathfrak{p}}X = 0$. \square

Consider as an example for T the stable module category $\text{StMod } kG$ of a finite group G with the canonical action of $R = H^*(G, k)$. We refer to [1, 2] for the discussion of two variations of the nucleus, namely the *group theoretic* and the *representation theoretic* nucleus. There it is shown that $\text{Loc}_T(\mathbb{1}) = T$ if and only if the centraliser of every element of order p in G is p -nilpotent and every block is either principal or semisimple, where p denotes the characteristic of the field k .

It is convenient to define for any class C of objects of T

$$\begin{aligned} C^\perp &= \{Y \in T \mid \text{Hom}_T^*(X, Y) = 0 \text{ for all } X \in C\}, \\ {}^\perp C &= \{X \in T \mid \text{Hom}_T^*(X, Y) = 0 \text{ for all } Y \in C\}. \end{aligned}$$

Now let $S = \text{Loc}_T(\mathbb{1})$. The *representation theoretic nucleus* is by definition

$$\bigcup_{X \in S^\perp \cap T^c} \text{supp}_R X.$$

Clearly, this is contained in the cohomological nucleus. It is a remarkable fact that the representation theoretic nucleus is non-empty if $S^\perp \neq 0$; this is proved in [1, 2]. Moreover, Question 13 of [7] asks whether $S = {}^\perp(S^\perp \cap T^e)$. Note that $S = {}^\perp(S^\perp)$ follows from general principles.

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